7.2 Laplace Transform of Initial-Value Problems

Solve, for example, mass-spring problem like

$$y'' + 4y = 8 \qquad y(0) = 0, \; y'(0) = 6$$

mass = 1   external (upward) force = 8

spring constant = 4   initial displacement = 0

                           "   velocity (up) = 6

basic idea: transform both sides, solve for $\mathcal{L}\{y\} = Y$

then inverse transform to find $y(t)$

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = ?$$

$$\mathcal{L}\{y'\} = \int_0^\infty y' e^{-st}\,dt = \lim_{a\to\infty} \int_0^a y' e^{-st}\,dt \qquad \begin{array}{ll} u = e^{-st} & dv = y'\,dt \\ du = -se^{-st}\,dt & v = y \end{array}$$

$$= \lim_{a\to\infty} \left(y(t)e^{-st} \Big|_0^a + s\int_0^a y e^{-st}\, dt \right)$$

$$= \lim_{a\to\infty} \left(y(a)\underbrace{e^{-sa}}_{\substack{\text{goes to } 0 \\ \text{so } s>0}} - y(0) \right) + s \underbrace{\int_0^\infty y e^{-st}\, dt}_{\mathcal{L}\{y\} = Y}$$

$$\boxed{\mathcal{L}\{y'\} = sY - y(0)} \quad s>0$$

$$\boxed{\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)}$$

$$\mathcal{L}\{y^{(n)}\} = s^nY - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

back to $y'' + 4y = 8 \quad y(0) = 0,\ y'(0) = 6$

$$\mathcal{L}\{y''\} + \mathcal{L}\{4y\} = \mathcal{L}\{8\}$$

$$s^2Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{6} + 4Y = \frac{8}{s}$$

Solve for Y

$$(s^2+4)Y = 6 + \frac{8}{s}$$

$$Y = \frac{6}{s^2+4} + \frac{8}{s(s^2+4)}$$

solution in s-domain

want solution in t-domain

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}} + \mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\}$$

directly from table: $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

$$\mathcal{L}^{-1}\left\{\frac{6}{s^2+2^2}\right\} = \mathcal{L}^{-1}\left\{3 \cdot \frac{2}{s^2+2^2}\right\} = 3\sin(2t)$$

$\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\}$ is a bit more complicated

no $\frac{1}{s(s^2+a^2)}$ on the table, but there is $\frac{1}{s}$, $\frac{1}{s^2+a^2}$

nice if $\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

partial fraction expansion

numerator is one degree lower than denominator

$$8 = A(s^2+4) + (Bs+C)s$$

$$0s^2 + 0s + 8 = (A+B)s^2 + Cs + 4A$$

$$A+B=0$$
$$C=0$$
$$4A = 8 \quad \text{so} \quad A=2, \; B=-2$$

$$\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+4}\right\}$$

$$= 2 - 2\cos(2t)$$

so, $\boxed{y(t) = 3\sin(2t) + 2 - 2\cos(2t)}$

another transform to look at:

we know $\mathcal{L}\{y'\} = sY - y(0)$

$$\mathcal{L}\{f'(t)\} = sF - f(0)$$

what is $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$?

let $g(t) = \int_0^t f(\tau)\,d\tau$

then, from calculus, $g'(t) = f(t)$

$$\mathcal{L}\{g'(t)\} = sG - \underbrace{g(0)}_{0} = \mathcal{L}\{f(t)\} = F$$

$$sG = F \qquad G = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$$

so, $s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = F$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F}{s} \longleftrightarrow \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{F}{s}\right\}$$

for example, $\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{8}{s^2+4}}{s}\right\}$ F find f then $\int_0^t f\,d\tau$

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\} = 4\sin(2t)$$

$$\int_0^t 4\sin(2\tau)\,d\tau = \cdots = -2\cos(2t) + 2$$

same as from partial fraction